UPPER BOUND FOR THE LEMPERT FUNCTION OF SMOOTH DOMAINS

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Abstract. An upper estimate for the Lempert function of any $C^{1+\varepsilon}$-smooth bounded domain in $\mathbb{C}^n$ is found in terms of the boundary distance.

1. Introduction

By $\mathbb{D}$ we denote the unit disc in $\mathbb{C}$. Let D be a domain in $\mathbb{C}^n$. Recall the definition of the Lempert function of D :

$$l_D(z,w)=\inf\{\alpha\in[0,1):\exists\varphi\in\mathcal{O}(\mathbb{D},D):\varphi(0)=z,\varphi(\alpha)=w\},$$

where $z,w\in D$. The Kobayashi pseudodistance k_D is the largest pseudodistance below $\tilde{k}_D=\tanh^{-1}l_D$.

Combining the proofs of Proposition 2.5 in [1] and Proposition 10.2.3 in [3], it follows that if D is bounded and $C^{1+\varepsilon}$-smooth, then there exists $c>0$ such that

$$(*)\quad 2k_D(z,w)\le\log\left(1+\frac{||z-w||}{d_D(z)}\right)+\log\left(1+\frac{||z-w||}{d_D(w)}\right)+c,$$

where d_D is the distance to ∂D. In [1] this inequality is applied for extension of proper holomorphic maps.

We point out that the assumption of smoothness is essential; the conclusion fails if D is a planar polygon (its boundary is Lipschitz).

On the other hand, one can show that if D is strongly pseudoconvex, then for any $\delta>0$ there exists a $c'>0$ such that

$$2k_D(z,w)\ge 2c_D(z,w)\ge-\log d_D(z)-\log d_D(w)-c'\quad\text{if } ||z-w||\ge\delta,$$

where $c_D(z,w)=\sup\{\tanh^{-1}|f(w)|:f\in\mathcal{O}(D,\mathbb{D}),f(z)=0\}$ is the Carathéodory distance of D. It is natural to ask whether the reverse

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inequality holds for $\tilde{k}_D$ (see [4]). Then the reverse inequality will imply $(*)$ if z and w are not close to one and the same boundary point. Such an estimate is equivalent to the following one.

Theorem 1. *Let $D \subset \mathbb{C}^n$ be a $C^{1+\varepsilon}$-smooth bounded domain. Then there is $c > 0$ such that for any $z, w \in D$,*

$$l_D(z,w) \le 1 - c d_D(z) d_D(w).$$

The following example may show that the smoothness assumption in the previous theorem is important.

Example 2. *Let $D \subset \mathbb{C}$ be the image of $\mathbb{D}$ under the map $z \to 2z + (1-z)\log(1-z)$. Then D is a C^1-smooth bounded domain and*

$$\lim_{\mathbb{R} \ni w \uparrow 2} \frac{1 - l_D(0,w)}{d_D(w)} = 0.$$

The main difficulty in the proof of Theorem 1 arises from the fact that, in general, l_D does not satisfy the triangle inequality. Therefore, we cannot localize this function, i.e. reduce the proof to the case when both arguments are near the boundary. On the other hand, the idea for the proof is clear: join two points (possibly on the boundary) by a suitable real-analytic curve in the domain and perturb a holomorphic extension of the curve in order to cover neighborhoods of these points. To control the perturbation, we shall need the upper bound in the following proposition.

Proposition 3. *Let (G_j) be a sequence of C^1-smooth simply connected bounded planar domains which tend to a bounded domain $G \subset \mathbb{C}$ in a sense that for any two open sets $K, L \subset \mathbb{C}$ with $K \Subset G \Subset L$ there exists an j_0 such that $K \subset G_j \subset L$ for any $j > j_0$. Let $\varepsilon > 0$. Assume that the following regularity properties hold: there is a neighborhood U of ∂G such that*

$$||\nabla r_j(z)|| \ge \varepsilon,\ ||\nabla r_j(z) - \nabla r_j(w)|| \le \varepsilon^{-1} ||z-w||^{\varepsilon}, \quad j \in \mathbb{N},\ z, w \in U,$$

where $r_j \in C^1(\mathbb{C})$ is a defining function of G_j (i.e. $G_j = \{z \in \mathbb{C} : r_j(z) < 0\}$). Then there is a $c > 0$ such that for any conformal map $f_j : \mathbb{D} \to G_j$ with $d_{G_j}(f_j(0)) \ge \varepsilon$ one has that

$$c^{-1} \le |f_j'(z)| \le c, \quad z \in \mathbb{D}.$$

Remark. Recall that if D is the inner domain of a Dini-smooth closed Jordan curve γ [1], then any conformal map $f : \mathbb{D} \to D$ extends to a diffeomorphism from $\overline{\mathbb{D}}$ to $\overline{D}$ (cf. Theorem 3.5 in [5]). Since $\tilde{k}_D = k_D$

[1]This means that $\int_0^1 \frac{\omega(t)}{t} dt < \infty$, where ω is the modulus of continuity of γ.

for any $D \subset \mathbb{C}$, it is not difficult to see that Theorem 1 holds for $n = 1$, if ∂D is a Dini-smooth curve near any boundary point.

2. Proofs

Proof of Example 2. The facts that the map is injective on $\mathbb{D}$ and G is a C^1-smooth domain may be found in [5], p. 46 (use e.g. Proposition 1.10). Let $\psi : D \to \mathbb{D}$ be the inverse map. Then

$$l_D(0, w) = l_{\mathbb{D}}(0, \psi(w)) = |\psi(w)|.$$

On the other hand, since D is C^1-smooth, it is not difficult to see that (use e.g. Proposition 2 in [2])

$$\lim_{w \to \partial D} |\psi'(w)| \frac{d_D(w)}{d_{\mathbb{D}}(\psi(w))} = 1.$$

Hence,

$$\lim_{\mathbb{R} \ni w \uparrow 2} \frac{1 - l_D(0, w)}{d_D(w)} = \lim_{\mathbb{R} \ni w \uparrow 2} |\psi'(w)| = 0.$$

Proof of Proposition 3. All the constant below will be independent of j and of the boundary points that appear.

Let $D \subset \mathbb{C}$ be a C^1-smooth bounded domain. For any point $a \in \partial D$ there is a $\theta_a \in \mathbb{R}$ such that if $\rho_a : z \to (z - a)e^{i\theta_a}$ and $D_a = \rho_a(D)$, then $x > 0$ ($z = x + iy$) is the inward normal to ∂D_a at 0. Moreover, for $\delta > 0$ put $G^i_\delta = \{z \in \mathbb{C} : |z| < 2\delta, x > |y|^{1+\delta}\}$ and $G^e_\delta = \mathbb{C} \setminus \overline{(-G^i_\delta)}$.

Using the assumptions of the proposition, we may shrink ε such that for any j and any $a \in \partial G_j$ one has that

$$G^i_\varepsilon =: G^i \subset G_{j,a} \subset G^e := G^e_\varepsilon.$$

We may smooth G^i and G^e at their angular points preserving these inclusions.

To prove the result, we shall need the following two estimates.

Estimate 1. There exists a $c_1 > 0$ such that

$$c_1^{-1} \le \kappa_{G_j}(z; 1) d_{G_j}(z) \le c_1, \quad z \in G_j,\ j \in \mathbb{N},$$

where κ_D denotes the Kobayashi-Royden metric of an arbitrary domain $D \subset \mathbb{C}$.[2]

Subproof. Fix $j \in \mathbb{N}$ and $z \in G_j$. First, we shall prove the lower bound. Let ψ be a conformal map from G^e to $\mathbb{D}_*$. Choose $a \in \partial G_j$ such that $d_{G_j}(z) = |z - a|$ and put $z_a = \rho_a(z)$. Then

$$\kappa_{G_j}(z; 1) \ge \kappa_{G^e}(z_a; 1) = \kappa_{\mathbb{D}_*}(\psi(z_a); \psi'(z_a)) \ge \kappa_{\mathbb{D}}(\psi(z_a); \psi'(z_a))$$

[2]Recall that $\kappa_D(z; X) = \inf\{\alpha \ge 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \alpha\varphi'(0) = X\}$ for any domain $D \subset \mathbb{C}^n$.

$$\geq c_3 d_{\mathbb{D}}^{-1}(\psi(z_a)) \geq c_4 d_{G_e}^{-1}(z_a) = c_4 d_{G_j}^{-1}(z)$$

(the constant c_3 is provided by the facts that ψ extends to a diffeomorphism from $\overline{G^e}$ to $\overline{\mathbb{D}}_*$ and that $\cup_{j=1}^{\infty} G_j$ is a bounded set).

Next, we prove the upper bound. Assume the contrary. Then we may find a sequence of points $z_j \in G_j$ (if necessary take an appropriate subsequence) such that $\kappa_{G_j}(z_j;1)d_{G_j}(z_j) \to \infty$. Since for any domains $G^2 \Subset G^1 \Subset G$ one has that

$$\kappa_{G_j}(z;1) \leq \kappa_{G^1}(z;1) \leq c(G^2), \quad j \gg 1, z \in G^2,$$

it follows that $d_{G_j}(z_j) \to 0$. Let $a_j \in \partial G_j$ be such that $d_{G_j}(z_j) = |z_j - a_j|$ and $z_{a_j} = \rho_{a_j}(z_j)$. Then $z_{a_j} \in G_i \cap (0, \varepsilon)$ for any $j \gg 1$. Similar to above, we get that

$$\kappa_{G_j}(z_j;1) \leq \kappa_{G^i}(z_{a_j};1) \leq c_5 d_{G^i}^{-1}(z_{a_j}) \leq c_6 z_{a_j}^{-1} = c_6 d_{G_j}^{-1}(z_j),$$

a contradiction.

Estimate 2. There is a $c_2 > 0$ such that if $d_{G_j}(w) \geq \varepsilon$, then

$$c_2^{-1} \leq \frac{1 - l_{G_j}(z,w)}{d_{G_j}(z)} \leq c_2.$$

Subproof. Following the proof of the lower bound in Estimate 1, we have

$$l_{G_j}(z,w) \geq l_{G^e}(z_a, w_a) = l_{\mathbb{D}}(\psi(z_a), \psi(w_a))$$
$$\geq 1 - c_7 d_{\mathbb{D}}(\psi(z_a)) \geq 1 - c_8 d_{G^e}(z_a) = 1 - c_8 d_{G_j}(z)$$

and the upper bound is proved (not using that $d_{G_j}(w) \geq \varepsilon$).

Next, we shall prove the lower bound. Since $d_{G_j}(w) \geq \varepsilon$, following the proof of the upper bound in Estimate 1, it is enough to show the bound when $z_a \in G^i \cap (0, \varepsilon)$. Then

$$k_{G_j}(z,w) = k_{G_{j,a}}(z_a, w_a) \leq k_{G_{j,a}}(z_a, \varepsilon) + k_{G_{j,a}}(\varepsilon, w_a)$$
$$\leq k_{G_{j,a}}(z_a, \varepsilon) + c_9 \leq k_{G^i}(z_a, \varepsilon) + c_9$$

(to see the second inequality, use G as above) and hence

$$1 - l_{G_j}(z,w) \geq c_{10}(1 - l_{G^i}(z_a, \varepsilon)) \geq c_{11} d_{G^i}(z_a) \geq c_{12} d_{G_j}(z).$$

Now, using both estimates, we shall prove the desired inequalities. Since $|z| = l_{\mathbb{D}}(0,z) = l_{G_j}(f_j(0), f_j(z))$, it follows by Estimate 2 that

$$c_2^{-1} \leq \frac{d_{G_j}(f_j(z))}{d_{\mathbb{D}}(z)} \leq c_2.$$

On the other hand,

$$(1 - |z|^2)^{-1} = \kappa_{\mathbb{D}}(z;1) = \kappa_{G_j}(f_j(z), f_j'(z)) = |f_j'(z)| \kappa_{G_j}(f_j(z);1).$$

Then Estimate 1 implies that

$$c_1^{-1} \le \frac{d_{G_j}(f_j(z))}{|f_j'(z)|d_{\mathbb{D}}(z)} \le 2c_1.$$

Hence $(2c_1c_2)^{-1} \le |f_j'(z)| \le c_1c_2$.

Proof of Theorem 1. By compactness, it is enough to prove the estimate when z and w are near two boundary points a and b (possible $a = b$), and when z lies in a compact subset of D, but w is near a boundary point b. We shall consider only the first case, because the second one is similar and even simpler.

Lemma 4. *There is a polynomial map $\varphi : \mathbb{C} \to \mathbb{C}^n$ such that*

$$\varphi((-1,1)) \subset D, \varphi(1) = a, \varphi(-1) = b, \varphi'(1) = -n_a, \varphi'(-1) = n_b,$$

where n_p is the inward normal vector to ∂D at p.

For completeness we shall prove this lemma at the end the paper.

Let $(u, v) \in T_a^{\mathbb{C}}\partial D \times T_b^{\mathbb{C}}\partial D$. Set

$$\varphi_{u,v}(\zeta) = \varphi(\zeta) + \left(\frac{\zeta+1}{2}\right)^2 u + \left(\frac{\zeta-1}{2}\right)^2 v, \quad \zeta \in \mathbb{C};$$

$$\Phi(\zeta_1, u, \zeta_2, v) = (\varphi_{u,v}(\zeta_1), \varphi_{u,v}(\zeta_2)).$$

Computing the Jacobian, it follows that $\Phi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is invertible at $(1, 0, -1, 0)$. Then for (z, w) in a neighborhood $U \subset \mathbb{C}^{2n}$ of (a, b) there are $\zeta_1(z, w), \zeta_2(z, w), u(z, w), v(z, w)$ such that

$$z = \varphi_{u(z,w),v(z,w)}(\zeta_1(z, w)),\ w = \varphi_{u(z,w),v(z,w)}(\zeta_2(z, w)).$$

Set $\psi_{z,w} = \varphi_{u(z,w),v(z,w)}$ and $G_{z,w} = \psi_{z,w}^{-1}(D)$. If r is a $C^{1+\varepsilon}$-smooth defining function of D, then $\rho_{z,w} = r \circ \psi_{z,w}$ is a defining function of $G_{z,w}$. Note that $(1, 0)$ and $(-1, 0)$ are the outward normal vectors to $\partial G_{a,b}$ at the points $A = (1, 0)$ and $B = (-1, 0)$, respectively. Then we may shrink U and find two squares $S_\delta(A)$ and $S_\delta(B)$ with side 2δ and centers at A and B, respectively, such that $\rho_{z,w}$ is close (in sense of the $C^{1+\varepsilon}$-norm) to the function $x - 1$ in $S_\delta(A)$ and to the function $-x - 1$ in $S_\delta(B)$. Since $[-1 + \delta/2, 1 - \delta/2] \Subset G_{a,b}$, we may find $\delta' > 0$ such that $R_{-\delta,\delta'} = \{\zeta : |x| < 1 - \delta/2, |y| < \delta'\} \Subset G_{a,b}$. We may shrink U such that $R_{-\delta,\delta'} \Subset G_{z,w}$ for any $(z, w) \in U$. Shrinking U further, we may assume that the curve $\gamma_{z,w} = \{\rho_{z,w} = 0\}$ intersects only once each of the horizontal line segments of length 2δ inside $S_\delta(A)$, and likewise for $S_\delta(B)$. Then $H_{z,w} = G_{z,w} \cap R_{2\delta,\delta'}$ is bounded by two horizontal line segments contained in $\{|y| = \delta'\}$ and by the curves $\gamma_{z,w} \cap S_\delta(A)$ and $\gamma_{z,w} \cap S_\delta(B)$; so $H_{z,w}$ fails to be smooth only at its four corners. We smooth $H_{z,w}$ such that it remains unchanged outside

of a $\delta'/2$ neighborhood of the corners, and that $H_{z,w}$ is close to $H_{a,b}$ (as before). Let $\eta_{z,w}:\mathbb{D}\to H_{z,w}$ be a conformal map with $\eta_{z,w}(0)=0$ and $\eta_{z,w}(p_{j,z,w})=\zeta_j(z,w)$, $j=1,2$. It extends to a diffeomorphism from $\overline{\mathbb{D}}$ to $\overline{H_{z,w}}$. By Proposition 3, reasoning by contradiction, we may shrink U and ε such that $\eta'_{z,w}$ are uniformly bounded from above. Setting $q_{j,z,w}=p_{j,z,w}/|p_{j,z,w}|$ and shrinking U once more, it follows by the mean-value inequality that $\rho_{z,w}(q_{j,z,w})\in\partial G_{z,w}$. The same inequality for $\theta_{z,w}=\psi_{z,w}\circ\eta_{z,w}\in\mathcal{O}(\mathbb{D},D)$ implies that

$$d_D(z)\le||\theta_{z,w}(q_{1,z,w})-\theta_{z,w}(p_{1,z,w})||\le Cd_{\mathbb{D}}(p_{1,z,w})$$

(since $\theta_{z,w}(q_{1,z,w})\in\partial D$) and similarly $d_D(w)\le Cd_{\mathbb{D}}(p_{2,z,w})$. Hence

$$1-l_D(z,w)\ge 1-\left|\frac{p_{1,z,w}-p_{2,z,w}}{1-p_{1,z,w}\overline{p_{2,z,w}}}\right|>\frac{d_{\mathbb{D}}(p_{j,z,w})d_{\mathbb{D}}(p_{j,z,w})}{2}\ge\frac{d_D(z)d_D(w)}{2C^2}.$$

Proof of Lemma 4. In the proof we will only assume that D is C^1-smooth near a and b. We start with a C^2-smooth curve $\tilde\varphi:[-1,1]\to\mathbb{C}^n$ such that

$$\tilde\varphi((-1,1))\subset D,\tilde\varphi(1)=a,\tilde\varphi(-1)=b,\tilde\varphi'(1)=-n_a,\tilde\varphi'(-1)=n_b.$$

Then for $\varepsilon>0$ choose a polynomial map $\varphi_\varepsilon:\mathbb{C}\to\mathbb{C}^n$ that agrees with $\tilde\varphi$ at ±1 up to order 1 and such that $||\varphi'_\varepsilon(t)-\varphi'(t)||<\varepsilon$ for any $t\in(-1,1)$. This map will do the job for any small ε.

Indeed, we shall show that there are $\varepsilon_1,\delta_1>0$ such that $\varphi_\varepsilon((1-\delta_1,1))\subset D$ for any $\varepsilon<\varepsilon_1$. Let r be a defining function of D which is C^1-smooth near a and b. Put $\rho_\varepsilon=r\circ\varphi_\varepsilon$. Then there exists a $\delta\in(0,1)$ such that

$$\rho_\varepsilon(1-t)=-2\int_{1-t}^1\mathrm{Re}\langle\partial r(\varphi_\varepsilon(s)),\overline{\varphi'_\varepsilon(s)}\rangle ds,\quad 0<t<\delta.$$

Shrinking δ, we may assume that $||2\partial r(z)+n_a||<1/4$ if $||z-a||<\delta$; in particular, $||\partial r(z)||<5/8$. Since

$$\varphi_\varepsilon(1-t)=a-\int_{1-t}^1\varphi'_\varepsilon(s)ds$$

and $||\varphi'_\varepsilon(s)-\tilde\varphi'(s)||<\varepsilon$, there are $\varepsilon_1,\delta_1>0$ such that $||\varphi_\varepsilon(s)-a||<\delta$ and $||\varphi'_\varepsilon(s)+n_a||<1/5$ if $1-s<\delta_1$ and $\varepsilon<\varepsilon_1$. Thus,

$$|1-2\mathrm{Re}\langle\partial r(\varphi_\varepsilon(s)),\overline{\varphi'_\varepsilon(s)}\rangle|\le2|\mathrm{Re}\langle\partial r(\varphi_\varepsilon(s)),\overline{\varphi'_\varepsilon(s)+n_a}\rangle|$$
$$+|\mathrm{Re}\langle2\partial r(\varphi_\varepsilon(s))+n_a,\overline{n_a}\rangle|<\frac{5}{4}.\frac{1}{5}+\frac{1}{4}.$$

Hence, $\mathrm{Re}\langle r(\varphi_\varepsilon(s)),\overline{\varphi'_\varepsilon(s)}\rangle>1/4$, which implies that $\rho_\varepsilon(1-t)<-t/2$, and we are done.

Similarly, there exist $\varepsilon_2, \delta_2 > 0$ such that $\varphi_\varepsilon((-1, -1+\delta_2)) \subset D$ for any $\varepsilon < \varepsilon_2$. Note that for $\delta_3 = \min\{\delta_1, \delta_2\}$ there is an $\varepsilon_3 > 0$ such that $\varphi_\varepsilon([-1+\delta_3, 1-\delta_3]) \subset D$ for any $\varepsilon < \varepsilon_3$. Therefore, any $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ does the job.

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